

Zero-Range Process with Open Boundaries

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We calculate the exact stationary distribution of the one-dimensional zero-range process with open boundaries for arbitrary bulk and boundary hopping rates. When such a distribution exists, the steady state has no correlations between sites and is uniquely characterized by a space-dependent fugacity which is a function of the boundary rates and the hopping asymmetry. For strong boundary drive the system has no stationary distribution. In systems which on a ring geometry allow for a condensation transition, a condensate develops at one or both boundary sites. On all other sites the particle distribution approaches a product measure with the finite critical density ρ_c . In systems which do not support condensation on a ring, strong boundary drive leads to a condensate at the boundary. However, in this case the local particle density in the interior exhibits a complex algebraic growth in time. We calculate the bulk and boundary growth exponents as a function of the system parameters.

KEY WORDS: Zero range process; open boundaries; invariant measure; hydrodynamical limit; condensation.

1. INTRODUCTION

The zero-range process (ZRP) has originally been introduced in 1970 by Spitzer⁽¹⁾ as a model system for interacting random walks, where particles on a lattice hop randomly to other neighboring sites. The hopping rates w_n depend only on the number of particles n at the departure site. This model has received renewed attention because of the occurrence of a condensation transition⁽²⁻⁶⁾ analogous to Bose-Einstein condensation and because of its close relationship with exclusion processes.⁽⁷⁾ Condensation phenomena are well-known in colloidal and granular systems (see ref. 8

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for a recent study making a connection with the zero-range process), but appear also in other contexts, such as socio-economics⁽⁹⁾ and biological systems⁽¹⁰⁾ as well as in traffic flow⁽¹¹⁾ and network theory.^(12,13) In the mapping of the ZRP to exclusion processes in one space dimension, condensation corresponds to phase separation. The ZRP has served for deriving a quantitative criterion for the existence of non-equilibrium phase separation⁽¹⁴⁾ in the otherwise not yet well-understood driven diffusive systems with two conservation laws. For the occurrence of condensation the dimensionality of the ZRP does not play a role and we shall consider only the one-dimensional (1d) case.

Most studies of the ZRP focus on periodic boundary conditions or the infinite system. Under certain conditions on the rates w_n (see below) the grand-canonical stationary distribution is a product measure, i.e. there are no correlations between different sites.⁽¹⁵⁾ In addition, an exact coarse-grained description of the *dynamics* is possible in this case in terms of a hydrodynamic equation for the particle density $\rho(x, t)$.^(16,17)

In zero-range processes for which the hopping rates w_n admit condensation, one finds that above a critical density ρ_c a finite fraction of all particles in the system accumulate at a randomly selected site, whereas all other sites have an average density ρ_c .^(5,18,19) The large scale dynamics of condensation has been studied in terms of a coarsening process.^(19,20) The steady-state and the dynamics of open systems which may admit condensation has not been addressed so far.

In the present work we consider a ZRP on an open chain with arbitrary hopping rates and boundary parameters. Particles are added and removed through the boundaries. In the interior of the system hopping may either be symmetric or biased in one direction. We calculate the exact steady-state distribution, when it exists, and find it to be a product measure. In order to study condensation phenomena in the open system we analyze in detail a particular but generic ZRP which admits condensation. In this model the hopping rates for large n take the form $w_n = 1 + b/n$. On a periodic lattice it is known that a condensation takes place at high densities when $b > 2$. We find that in an open system and for a weak boundary drive the model evolves to a non-critical steady-state. On the other hand, if the boundary drive is sufficiently strong, the system may develop a condensate on one or both of its boundary sites even for $b < 2$. The number of particles in the condensate grows linearly in time due to the strong boundary drive. In this case the interior of the system may either (a) reach a sub-critical steady-state, (b) reach a critical steady-state, or (c) it may evolve such that the local particle density exhibits a complex algebraic growth in time. Which behavior is actually realized depends on the

boundary rates, on the parameter b , and on the asymmetry in the hopping rates.

The paper is organized as follows. In Section 2 we define the ZRP with open boundaries, and give some examples of possible hopping rates w_n . The exact steady-state distribution for general boundary parameters and rates w_n is derived in Section 3. In Section 4 we consider the behavior of finite systems for totally asymmetric, partially asymmetric and symmetric hopping rates. The long-time temporal behavior of local densities is obtained, and supporting numerical simulations are presented. The long-time behavior of bulk densities in an infinite system is exactly obtained using a hydrodynamical approach in Section 5. Finally, we present our summary and conclusions in Section 6.

2. ZERO-RANGE PROCESSES WITH OPEN BOUNDARIES

The ZRP on an open 1d lattice with L sites is defined as follows. Each site k may be occupied by an arbitrary number n of particles. In the bulk a particle at site k (say, the topmost of n particles) hops randomly (with exponentially distributed waiting time) with rate $p w_n$ to the right and with rate $q w_n$ to the left. Without loss of generality, we take throughout this paper $p \geq q$, so that particles in the bulk are driven to the right. At the boundaries these rules are modified. At site 1 a particle is injected with rate α , hops to the right with rate $p w_n$, and is removed with rate γw_n . At site L a particle is injected with rate δ , hops to the left with rate $q w_n$, and is removed with rate βw_n (see Fig. 1).

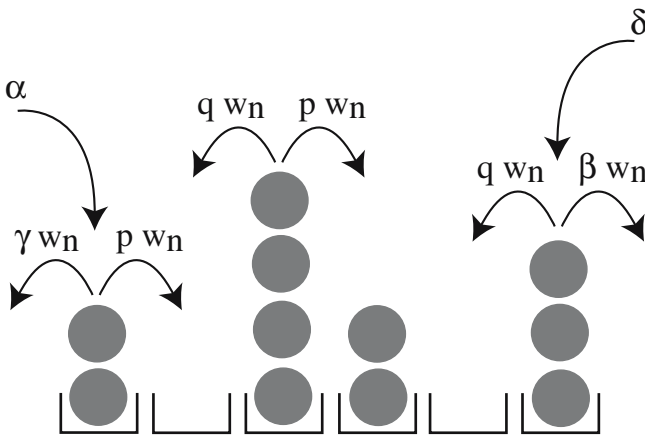


Fig. 1. Graphic representation of the ZRP model with open boundaries.

Examples for such processes include the trivial case of non-interacting particles ($w_n = n$) and the pure chipping process ($w_n = 1$).⁽²¹⁾ In the mapping to exclusion processes where the particle occupation number becomes the interparticle distance, the chipping model defined on a ring maps onto the simple exclusion process which is integrable and can be solved by the Bethe ansatz, combinatorial methods, recursion relations and matrix product methods.^(22,23) A largely unexplored but interesting integrable model with rates $w_n = (1 - q^n)/(1 - q)$ that interpolates between these two cases has been found in ref. 24. The truncated chipping process with $w_n = 1$ for $1 \leq n \leq K$ and $w_n = \infty$ for $n > K$ maps onto the drop-push model⁽²⁵⁾ which is also integrable.⁽²⁶⁾ For $w_1 = w$ and $w_n = 1$ for $n \geq 2$ one obtains the version of the non-integrable KLS model⁽²⁷⁾ that has been introduced in ref. 28 as a toy model for traffic flow with a non-symmetric current density relation. A parallel updating version of this model was found to correspond to a broader class of traffic models⁽²⁹⁾ as well as an integrable family of ZRP's⁽³⁰⁾ which includes the asymmetric avalanche process.⁽³¹⁾

In a similar mapping to exclusion processes the family of models with hopping rate

$$w_n = \left[1 + \frac{b'}{n} \right]^{-1}, \quad (1)$$

describes the diffusion of interacting rods on the real line.⁽³²⁾ In this model the occupation number n of the zero-range process maps to the (integer) distance of rods of arbitrary (integer) length which have long-range interaction with rates given by Eq. (1). In the continuum limit the length of these rods and the distance n become finite real numbers. The condensation model^(4,5,14,18-20) with

$$w_n = 1 + \frac{b}{n} \quad (2)$$

is a generic model that exhibits the condensation phenomenon described in the introduction for $b > 2$. Both models are non-integrable and hence alternative tools must be employed for deriving information about their dynamical behavior. We remind the reader that this model is generic in the sense that it represents the complete family of models with rates of the form $w_n = 1 + b/n + O(n^{-s})$ where $s > 1$. Other ZRP's which exhibit condensation are defined by the rates $w_n = 1 + b/n^\sigma$ with $0 < \sigma < 1$ and $b > 0$ ⁽¹⁴⁾ or by rates approaching zero.⁽⁶⁾

3. STATIONARY DISTRIBUTION

A state of the model at time t may be defined through a probability measure P_n on the set of all configurations $n = (n_1, n_2, \dots, n_L)$, $n_k \in \mathbb{N}$. Here n_k is the number of particles on site k . To calculate the stationary distribution it is convenient to represent the generator H of this process in terms of the quantum Hamiltonian formalism⁽²³⁾ where one assigns a basis vector $|n\rangle$ of the vector space $(\mathbb{C}^\infty)^{\otimes L}$ to each configuration and the probability vector is defined by $|P\rangle = \sum_n P_n |n\rangle$. It is normalized such that $\langle s|P\rangle = 1$ where $\langle s| = \sum_n \langle n|$ and $\langle n|n'\rangle = \delta_{n,n'}$. The time evolution described above is given by the master equation

$$\frac{d}{dt} |P(t)\rangle = -H |P(t)\rangle, \tag{3}$$

through the ‘‘quantum Hamiltonian’’ H . This operator has off-diagonal matrix elements $H_{n,n'}$ which are the hopping rates between configurations n, n' and complementary diagonal elements to preserve conservation of probability.

Since we have only nearest neighbor exchange processes the Hamiltonian in (3) can be written as

$$H = h_1 + h_L + \sum_{k=1}^{L-1} h_{k,k+1}, \tag{4}$$

where $h_{k,k+1}$ acts nontrivially only on sites k and $k + 1$ (corresponding to hopping) while h_1, h_L generates the boundary processes specified above. For the ZRP we define the infinite-dimensional particle creation and annihilation matrices

$$a^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad a^- = \begin{pmatrix} 0 & w_1 & 0 & 0 & \dots \\ 0 & 0 & w_2 & 0 & \dots \\ 0 & 0 & 0 & w_3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{5}$$

as well as the diagonal matrix d with elements $d_{i,j} = w_i \delta_{i,j}$. With these matrices we have

$$-h_{k,k+1} = p(a_k^- a_{k+1}^+ - d_k) + q(a_k^+ a_{k+1}^- - d_{k+1}) \tag{6}$$

and

$$-h_1 = \alpha(a_1^+ - 1) + \gamma(a_1^- - d_1), \quad -h_L = \delta(a_L^+ - 1) + \beta(a_L^- - d_L). \quad (7)$$

The “ground state” of H has eigenvalue 0. The corresponding right eigenvector is the stationary distribution which we wish to calculate.

Guided by the grand-canonical stationary distribution of the periodic system we consider the grand-canonical single-site particle distribution where the probability to find n particles on site k is given by

$$P^*(n_k = n) = \frac{z_k^n}{Z_k} \prod_{i=1}^n w_i^{-1}. \quad (8)$$

Here the empty product $n=0$ is defined to be equal to 1 and Z_k is the local analogue of the grand-canonical partition function

$$Z_k \equiv Z(z_k) = \sum_{n=0}^{\infty} z_k^n \prod_{i=1}^n w_i^{-1}. \quad (9)$$

The corresponding probability vector $|P_k^*\rangle$ with the components $P^*(n_k = n)$ satisfies

$$a^+ |P_k^*\rangle = z_k^{-1} d |P_k^*\rangle, \quad a^- |P_k^*\rangle = z_k |P_k^*\rangle. \quad (10)$$

The proof of this property is by straightforward calculation.

As an ansatz for calculating the stationary distribution we take the L -site product measure with the one-site marginals (8) which is given by the tensor product

$$|P^*\rangle = |P_1^*\rangle \otimes |P_2^*\rangle \otimes \cdots \otimes |P_L^*\rangle \quad (11)$$

and according to (10) satisfies

$$-H |P^*\rangle = \left[\sum_{k=1}^{L-1} (pz_k - qz_{k+1})(z_{k+1}^{-1} d_{k+1} - z_k^{-1} d_k) + (\alpha - \gamma z_1)(z_1^{-1} d_1 - 1) + (\delta - \beta z_L)(z_L^{-1} d_L - 1) \right] |P^*\rangle. \quad (12)$$

The uncorrelated particle distribution (11) is stationary if and only if all terms on the right hand side of this equation cancel. It is not difficult to see that this is satisfied for the following stationarity conditions on the fugacities z_k

$$pz_k - qz_{k+1} = \alpha - \gamma z_1 = \beta z_L - \delta \equiv c. \tag{13}$$

The quantity c is the stationary current.

This recursion relation has the unique solution

$$z_k = \frac{[(\alpha + \delta)(p - q) - \alpha\beta + \gamma\delta] \left(\frac{p}{q}\right)^{k-1} - \gamma\delta + \alpha\beta \left(\frac{p}{q}\right)^{L-1}}{\gamma(p - q - \beta) + \beta(p - q + \gamma) \left(\frac{p}{q}\right)^{L-1}} \tag{14}$$

and the current is given by

$$c = (p - q) \frac{-\gamma\delta + \alpha\beta \left(\frac{p}{q}\right)^{L-1}}{\gamma(p - q - \beta) + \beta(p - q + \gamma) \left(\frac{p}{q}\right)^{L-1}}. \tag{15}$$

Our process is defined on a finite lattice. With rates $w_n > 0$ for all $n \geq 1$ the process is therefore ergodic and hence the invariant product measure is, when it exists, unique. The local fugacities z_k are completely specified by the hopping asymmetry and the boundary parameters. The stationary density profile follows from the fugacity profile through the standard relation

$$\rho_k = z_k \frac{\partial}{\partial z_k} \ln Z_k, \tag{16}$$

where $Z_k = Z(z_k)$ is determined by the bulk hopping rules, as given in (9).

We remark that up to corrections exponentially small in system size L , the bulk fugacity is a constant

$$z_{\text{eff}} = \frac{\alpha}{p - q + \gamma}, \tag{17}$$

that depends only on the *left* boundary rates. This is in agreement with the observation⁽¹⁷⁾ for special boundary parameters and can be explained in terms of the more general theory of boundary-induced phase transitions.⁽³³⁾

Some special cases of (14) deserve mentioning.

(1) Setting the boundary extraction rates equal to the corresponding bulk jump rates, i.e., $\beta = p$, $\gamma = q$, and defining reservoir fugacities $z_{r,l}$ by $\alpha = pz_l$, $\delta = qz_r$, the expression (14) reduces to

$$z_k = \frac{(z_l - z_r) \left(\frac{p}{q}\right)^{k-1} + z_r - z_l \left(\frac{p}{q}\right)^{L-1}}{1 - \left(\frac{p}{q}\right)^{L-1}}. \tag{18}$$

This dynamics has a natural interpretation as coupling the system to boundary reservoirs with fugacities z_r, z_l , respectively. This case has been considered in refs. 15, 17, 34, 35 for $p = q$. The general solution (14) for arbitrary boundary rates appears to be a new result for the ZRP.

(2) For

$$\gamma\delta = \alpha\beta \left(\frac{p}{q}\right)^{L-1} \tag{19}$$

the current vanishes and the system is in thermal equilibrium. For the symmetric case $p = q$ this implies constant fugacities z_k .

(3) For symmetric hopping $p = q = 1$ the fugacity profile is generally linear

$$z_k = \frac{\alpha + \delta + \alpha\beta(L - 1) - (\alpha\beta - \gamma\delta)(k - 1)}{\beta + \gamma + \beta\gamma(L - 1)} \tag{20}$$

with a current

$$c = \frac{\alpha\beta - \gamma\delta}{\beta + \gamma + \beta\gamma(L - 1)}. \tag{21}$$

Notice that the linear fugacity profile does not imply a linear density profile except in the very special case of non-interacting particles where $z_k = \rho_k$.

(4) In the totally asymmetric case, $q = 0$, we find

$$z_k = \frac{\alpha}{p + \gamma} \equiv z \quad \text{for } k \neq L, \tag{22}$$

$$z_L = \frac{(\alpha + \delta)p + \gamma\delta}{\beta(p + \gamma)}. \tag{23}$$

The current is given by $c = \rho z$.

In the considerations above we have tacitly assumed that the local partition function Z_k exists for all k . Since for suitable choices of boundary parameters the local fugacities at the boundary may take arbitrary values this assumption implies an infinite radius of convergence of $Z_k = Z(z_k)$ which is not the case in the models described above. This raises the question of the long-time behaviour of the ZRP for strong boundary drive, i.e., rates which drive the boundary fugacities out the radius of convergence of Z . This is studied in detail in the following sections for the condensation model (2). The rod model (1) can be regarded as a generic ZRP without bulk condensation transition, but finite radius of convergence for Z . Where appropriate we compare its behaviour with that of the condensation model.

4. CONDENSATION MODEL — STEADY STATE AND DYNAMICS NEAR THE BOUNDARY

In this section we consider in some detail the long-time behavior of the condensation model (2). For this model the local fugacity z_k has to satisfy $z_k \leq 1$ in the steady state. On a ring geometry, the model exhibits a condensation for $b > 2$ at high density. We first analyze the totally asymmetric case, where particles can only hop to the right. The cases of partial asymmetry and symmetric hopping are then treated.

4.1. Totally Asymmetric Hopping

For the totally asymmetric case we take $q = \gamma = \delta = 0$. For the normalization of time we set $p = 1$. The exact steady-state solution (14) yields

$$z_k = \alpha \equiv z \text{ for } k \neq L, \quad (24)$$

$$z_L = \frac{\alpha}{\beta} \quad (25)$$

and the current is given by $c = z$. Since for this model z has to satisfy $z \leq 1$, the steady state (24, 25) is valid only for $\alpha \leq 1$ and $\beta \geq \alpha$. In this case the single-site steady-state distribution is given, for large n , by $P^*(n_k = n) \sim z^n / n^b$.

We now proceed to discuss the dynamical behavior of the model in the case that a stationary state does not exist. For $\alpha > 1$ the following picture emerges.

Site 1. On site 1 particles are deposited randomly with rate $\alpha > 1$ and are removed by hopping to site 2 with rate $1 + b/n_1$. Hence the occupation number performs a simple biased random walk on the set n_1 of

positive integers with drift $\alpha - 1 - b/n_1$ which is positive for any $n_1 > b/(\alpha - 1)$. Such a random walk is non-recurrent and reaches asymptotically the mean velocity $v = \alpha - 1$. Hence the mean particle number $N_1(t) = \langle n_1(t) \rangle$ on site 1 grows asymptotically linearly

$$N_1(t) \sim (\alpha - 1)t. \quad (26)$$

Boundary Sites $k > 1$. We extend the random walk picture (which is strictly valid for site 1) to site 2. On site 2 particles are injected (by hopping from site 1) with rate $1 + b/n_1$ and are removed with rate $1 + b/n_2$. Since n_1 increases in time on average the input rate approaches 1 and the occupation number at site 2 performs a biased random walk with hopping rate 1 to the right and rate $1 + b/n_2$ to the left. Whether this random walk is recurrent depends on b . The asymptotic behavior has been analyzed in ref. 36. We merely quote the result:

$$N_2(t) \sim \begin{cases} t^{1/2} & b < 1, \\ t^{1/2}/\ln t & b = 1, \\ t^{1-b/2} & 1 < b < 2, \\ \ln t & b = 2, \\ \rho^* & b > 2. \end{cases} \quad (27)$$

The constant

$$\rho^* = \frac{1}{b-2} \quad (28)$$

is the critical density of the condensation model.⁽⁴⁾ It is approached with a power law correction $t^{1-b/2}$. By applying this random walk picture to further neighboring sites, and assuming scaling, it has been shown that neighboring boundary sites behave asymptotically in the same fashion.⁽³⁶⁾ Similar analysis shows that the square-root increase of the particle density occurs also for the model (1) for all values of its interaction parameter b' .

To test the validity of the random walk picture to sites beyond $k=2$, we carried out numerical simulations of a totally asymmetric model with $L=5$. We first note that since the hopping is totally asymmetric the time evolution of the system up to site k is independent of what happens at sites to the right of k . In particular, the dynamics on all sites $k < L$ is independent of β . Hence, in order to study boundary layers it is sufficient to simulate very small systems of only a few sites. In Fig. 2 we present the long-time behavior of the occupation of sites $k=1-4$ for $\alpha=2$ and

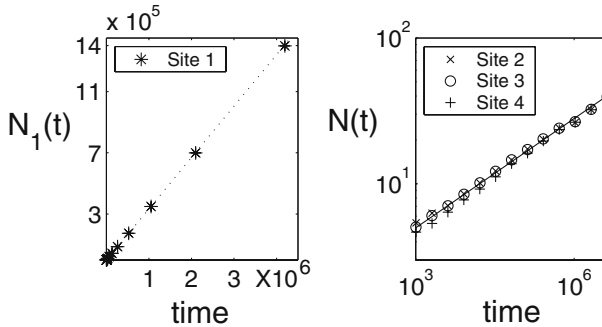


Fig. 2. Temporal evolution of local densities, as obtained from numerical simulations of the totally asymmetric condensation model, with $b = 3/2$, and $\alpha = 2$. The solid line corresponds to the expected growth law $t^{1/4}$, and the dotted line corresponds to linear growth.

$b = 3/2$. It is readily seen that while site 1 grows linearly in time, sites 2–4 grow with the expected power law $t^{1/4}$.

Bulk Sites $k \gg 1$. The picture of the simple random walk becomes increasingly inaccurate as one enters into the bulk of the system, since injection events onto a site become increasingly correlated in time. This violates the random walk assumption and makes the previous analysis invalid in this case. The temporal behavior of bulk sites $k \gg 1$ can be treated exactly by using a hydrodynamic approach, which yields the behavior of bulk sites in the long-time limit. This analysis, carried out in Section 5, shows a different dynamical behavior in the bulk. Notice, however, that for any *finite* system all “bulk sites” have finite distance from the boundary and hence behave asymptotically like the boundary sites.

Site L . Since the motion of particles on site k is independent of the motion on sites to their right we expect the bulk result to be asymptotically valid on all sites up to site $L - 1$, i.e., there is no right boundary layer with yet another set of growth exponents. On site L the following picture emerges. There is an asymptotic incoming flux $c = 1$ and an exit rate $\beta(1 + b/n_L)$. For $\beta = 1$ the boundary site behaves like a bulk site and we obtain the bulk growth exponent. For $\beta < 1$ the outgoing flux does not compensate the incoming flux which yields asymptotically linear growth $\rho_L(t) = (1 - \beta)t$. For $\beta > 1$ a finite stationary chemical potential $z_L = 1/\beta$ is approached.

In Fig. 3 we present simulation results for the long-time dynamics of the occupation of site $k = L = 5$. Depending on the value of β the occupation number either grows linearly ($\beta < 1$), grows with the same power law

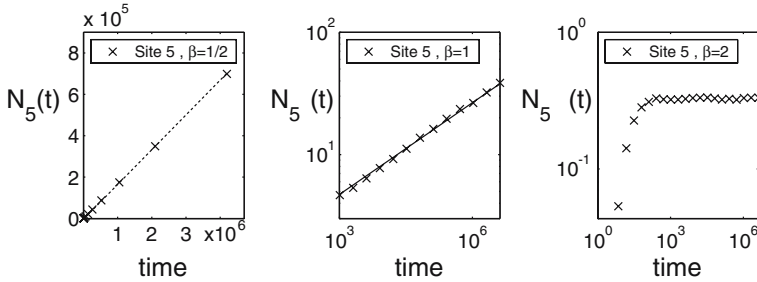


Fig. 3. Temporal evolution of local density at the rightmost site, as obtained from numerical simulations of the totally asymmetric condensation model, with $b=3/2$, $\alpha=2$ and $\beta=1/2, 1, 2$. The solid line corresponds to the expected growth law $t^{1/4}$, and the dotted line corresponds to linear growth.

as the bulk ($\beta=1$), or approaches a finite density ($\beta > 1$), as expected from the above discussion.

To complete the discussion of the totally asymmetric case we remark that for a subcritical left boundary $\alpha < 1$ but supercritical $\beta < \alpha$ the bulk of the system becomes stationary with fugacity $z_{\text{bulk}}=\alpha$. On site L a condensate develops with linearly increasing particle density $\rho_L(t) \sim (\alpha - \beta)t$.

4.2. Partially Asymmetric Hopping

We now analyze the dynamical behavior of the partially asymmetric model in the case where no stationary state exists. We start by considering the case where the rates at both boundaries are such that the occupation of the two boundary sites increase linearly with time. This takes place for $\alpha - \gamma < p - q < \beta - \delta$. Here sites $k=1$ and $k=L$ act as reservoirs for the rest of the system. The effective rates at which the reservoirs exchange particles with the system are $\alpha_{\text{eff}} = \beta_{\text{eff}} = p$ and $\gamma_{\text{eff}} = \delta_{\text{eff}} = q$. The fugacity (14) at sites $k=2, \dots, L-1$ thus becomes $z_k=1$. The asymptotic temporal behavior (27) holds also in this case.

The picture changes qualitatively if only one boundary fugacity does not exist. Suppose first that this happens at site 1. In this case the density on site 1 will increase linearly with time, as in the previous totally asymmetric case. This happens when $\alpha - \gamma > p - q$ and $\beta - \delta > p - q$. As before, site 1 acts as a reservoir for the rest of the system, with effective rates $\alpha_{\text{eff}} = p$, $\gamma_{\text{eff}} = q$. Sites $k=2, \dots, L-1$ are thus stationary, with the fugacity given by (14). The fugacity at sites away from the right boundary approaches 1, with a deviation exponentially small in the system size (see Fig. 4). Hence, starting from an empty initial state one expects algebraic

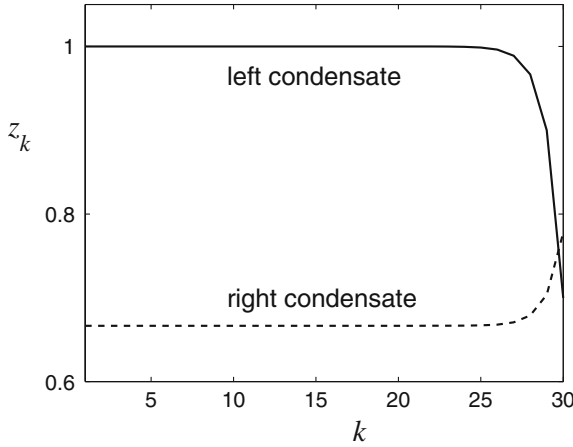


Fig. 4. The fugacity profile (14) for the partially asymmetric case, with $b = 3/2$, $p = 3/4$ and $q = 1/4$. Solid line corresponds to a left condensate ($\alpha = 3/4, \beta = 1, \gamma = 1/4, \delta = 1/5$) and dashed line corresponds to a right condensate ($\alpha = 1/2, \beta = 3/4, \gamma = 1/4, \delta = 1/4$).

growth of the local density until, after a long crossover time, stationarity is reached.

If, on the other hand, the right boundary rates drive site L out of equilibrium, then the density on site L increases linearly in time. This happens when $\alpha - \gamma < p - q$ and $\beta - \delta < p - q$. Site L acts effectively as a boundary reservoir with $\beta_{\text{eff}} = p$ and $\delta_{\text{eff}} = q$. As in the preceding case the system becomes stationary, but with a finite (independent of system size) deviation of the bulk fugacity from the critical value $z = 1$. Both in the bulk and at the left boundary the density approaches the finite value dictated by the left boundary fugacity (see Fig. 4).

Finally, for $\beta - \delta < p - q < \alpha - \gamma$ both boundary sites have finite fugacity, and the system reaches a non-critical steady state, as given by (14).

4.3. Symmetric Hopping

The analysis of symmetric hopping ($p = q = 1$) follows very closely that of the partially asymmetric case, except that here the fugacity profile is linear rather than exponential. In particular, if only one boundary is driven out of equilibrium, a condensate appears at that boundary with a particle density which increases linearly with time. This site, say site 1, acts as a reservoir with $\alpha_{\text{eff}} = \gamma_{\text{eff}} = 1$. The fugacity profile decreases linearly from $1 - O(1/L)$ at the left boundary to $\delta/\beta + O(1/L)$ at the right boundary, as given by (20).

When both boundary sites are supercritical they act as reservoirs with effective rates $\alpha_{\text{eff}} = \beta_{\text{eff}} = \gamma_{\text{eff}} = \delta_{\text{eff}} = 1$, yielding $z_k = 1$ throughout the system. This is also the case relevant for studying the fluid-mediated interaction of probe particles in two-species systems.⁽³⁶⁾ We expect similar boundary growth laws as in the asymmetric case. This is because the incoming flux from the direction of the bulk is compensated by the outgoing flux at the boundary and one has an effective random picture with the same rates as above. However, corrections to scaling are expected to be larger as current correlations are now stronger due to hopping contributions from the bulk. Moreover, as opposed to the asymmetric case, in a semi-infinite system both boundaries have the same growth exponents.

5. HYDRODYNAMICS — EXACT ANALYSIS OF DYNAMICS IN THE BULK

In order to analyze the time evolution of sites far away from the boundaries we consider the hydrodynamic limit of the ZRP model. The coarse-grained time evolution of the density profile starting from a non-stationary initial profile can be determined, by adapting standard arguments,^(16,37) from the continuum limit of the lattice continuity equation. This equation reads

$$\frac{d}{dt} \rho_k = c_{k-1} - c_k \quad (29)$$

with the local current

$$c_k = pz_k - qz_{k+1} \quad (30)$$

and z expressed in terms of ρ . Together with an appropriate choice of constant boundary fugacities the solution is uniquely determined.⁽¹⁷⁾

For the driven system one obtains under Eulerian scaling (lattice constant $a \rightarrow 0$, $t \rightarrow t/a$, system length fixed)

$$\frac{\partial}{\partial t} \rho(x, t) = -(p - q) \frac{\partial}{\partial x} z(x, t) + a(p + q)/2 \frac{\partial^2}{\partial x^2} z(x, t). \quad (31)$$

This time evolution equation follows from (29) by a Taylor expansion of the space-dependence, keeping as regularization the lowest order term in the lattice constant. Notice that this infinitesimal viscosity term does not represent the true lowest order correction to the hydrodynamic equation (which is not the object of our investigation), but rather serves as a

regularization that selects the physical solution of the otherwise ill-posed initial value problem with fixed boundary fugacities. It takes care of a proper description of discontinuities which may arise in the form of shocks or a boundary discontinuity. Indeed, in the large-time limit the density approaches a constant given by the left boundary fugacity, with a jump discontinuity at the right boundary.⁽¹⁷⁾ This is in agreement with the exact result derived in Section 4.2. We stress that (31) provides an exact description of the density evolution under Eulerian scaling. It is not a continuum approximation involving a mean field or other uncontrolled assumption, even though for this model a standard mean-field analysis which neglects all correlations would lead to the same result. The simple form of (31) originates in local stationarity. The absence of noise is due to Eulerian scaling, i.e., the effects of noise appear on finer scales. For a recent rigorous discussion of the hydrodynamic limit of stochastic particle systems, (see refs. 16, 37 and 38).

Consider a semi-infinite system which is initially empty and has constant density at the left boundary. This boundary condition induces a rarefaction wave entering the bulk which can be constructed using the method of characteristics. The speed v_0 of the wave front is given by the zero-density characteristic of (31) which is the average speed $v_0 = (p - q)w_1$ of a single particle. In light of the results of the previous sections we are particularly interested in the case where the left boundary fugacity is equal to 1. For $b < 2$ in the condensation model [and for any b' in the rod model (1)] this corresponds to an infinite boundary density. On the other hand, for $b > 2$ the corresponding boundary density is ρ_c (28). Therefore we are searching for a scaling solution in terms of the scaling variable $u = x/(v_0 t)$ such that $\rho(u) = 0$ for $u \geq 1$ while $\rho = \rho_c$ or $\rho = \infty$ at the left boundary. On physical grounds the solution has to be continuous as no shock discontinuities can develop for the initial state (empty lattice) under consideration. Under these conditions (31) can be integrated straightforwardly by setting $a = 0$ and one finds the implicit representation

$$\frac{dz}{d\rho} = u w_1. \quad (32)$$

This is the solution within the interval $v_1 t \leq x \leq v_0 t$. Here

$$v_z = (p - q) \frac{dz}{d\rho} \quad (33)$$

is the collective velocity of the lattice gas which plays the role of the speed of the characteristics for the hydrodynamic equation (31). Outside

this interval one has $z=0$ for $x \geq v_0 t$ and $z=1$ for $0 \leq x \leq v_1 t$. We now analyze this solution in terms of ρ .

For the model (1) one has $Z = 1/(1 - z)^{b'+1}$ which yields the fugacity-density relation $z = \rho/(b' + 1 + \rho)$. Hence

$$\rho(u) = (b' + 1) \left(\sqrt{\frac{1}{u}} - 1 \right). \tag{34}$$

with $v_0 = (p - q)/(b' + 1)$ and $v_1 = 0$ for all b' . At any fixed x the bulk density increases algebraically

$$\rho_{\text{bulk}}(t) \sim t^{1/2} \tag{35}$$

for any $b' > 0$.

For the condensation model the local partition is the hypergeometric function

$$Z = {}_2F_1(1, 1; 1 + b; z), \tag{36}$$

which does not admit an explicit representation of z as a function of ρ . However, as the fugacity is an increasing function in time approaching 1 we may analyze its asymptotic behavior by expanding the hypergeometric function around $z = 1$.⁽¹⁹⁾

$b < 2$. Here the left boundary fugacity $z = 1$ corresponds to infinite left boundary density. Moreover $v_1 = 0$, therefore the solution (32) describes the density profile in the interval $0 \leq u \leq 1$. This enables us to consider fixed x and study the long time limit. In order to analyze the small u behavior (i.e. z close to 1 and ρ large) one has to distinguish two domains.⁽¹⁹⁾ For $b < 1$ one has $\rho(z) \propto z/(1 - z)$ for large ρ while for $1 < b < 2$ one finds $\rho(z) \propto z/(1 - z)^{2-b}$. As a function of u we make the ansatz $\rho \propto 1/u^\kappa$ for the large t (i.e. small u) asymptotics. Inserting this into the differential equation (31) yields a consistent solution with

$$\rho_{\text{bulk}}(t) \sim t^\kappa \tag{37}$$

only for

$$\kappa = \begin{cases} \frac{1}{2} & \text{for } b < 1, \\ \frac{2-b}{3-b} & \text{for } 1 < b < 2 \end{cases} \tag{38}$$

We conclude that the bulk density increases algebraically with the universal diffusive exponent $1/2$ for $b < 1$ and with a non-universal b -dependent exponent in the range $1 < b < 2$ of the condensation model. At $b = 1, 2$ there are logarithmic corrections which we do not discuss further.

For symmetric hopping one describes the density dynamics under diffusive scaling $t \rightarrow t/a^2$ and obtains

$$\frac{\partial}{\partial t} \rho(x, t) = p \frac{\partial^2}{\partial x^2} z(x, t), \quad (39)$$

which needs no further regularization. Repeating the previous analysis one finds the same bulk growth exponents as for the asymmetric hopping model.

$b > 2$. In the condensation regime the boundary density corresponding to $z = 1$ is ρ_c (28). Again two different regimes are found from the asymptotic analysis of the hypergeometric function. For $b < 3$ one has $v_1 = 0$ and repeating the same analysis as for $1 < b < 2$ shows that at fixed x the density approaches ρ_c with a power law correction with the same exponent $(2-b)/(3-b)$ as before. This is analogous to the behavior in the boundary sites analysed in Section 4, but the exponent is different. For $b > 3$ one finds for the collective velocity⁽¹⁹⁾

$$v_1 = (p - q) \frac{(b-3)^2(b-2)^2}{(b-1)^2} > 0. \quad (40)$$

Hence an analysis of the long-time behaviour at fixed x is not meaningful. A domain with constant $\rho = \rho_c$ spreads into the system, with a front speed $v = v_1$. This front is preceded by the rarefaction wave (32) for $v_1 t < x < v_0 t$. This behaviour as a function of b is unexpected as usually changes in the rarefaction wave of this type are caused by changing the boundary density rather than an interaction parameter of the driven system.

We stress that there are two questions that cannot be answered by the hydrodynamic analysis given above. First we apply Eulerian or diffusive scaling, respectively. This leaves generally open what happens in any semi-infinite lattice system at finite lattice distance from the boundaries or in a finite system. Any deviation from the results given above which decays on lattice scale as one approaches the bulk cannot be detected within the hydrodynamic description. Boundary layers, which have been analyzed in Section 4 and found to have an interesting microscopic structure, would under scaling at best appear as a structureless boundary discontinuity.

Secondly, the radius of convergence of Z is 1 and forcing the boundary fugacities z_1 or z_L to be larger than 1 implies a breakdown of the assumption of local stationarity underlying the hydrodynamic description, at least in the vicinity of the boundaries. In particular, the hydrodynamic approach is not applicable for analyzing the condensation regime $b > 2$ if the boundary density exceeds the critical density of the bulk. However, this regime may be analyzed by the random-walk approach discussed in Section 4.

6. CONCLUSIONS

In this paper we studied the dynamical behavior of the zero-range process with open boundary conditions for arbitrary bulk and boundary rates. It is found that for a weak boundary drive the model reaches a steady state. The exact steady-state distribution is calculated and is shown to be a product measure characterized by site-dependent fugacities. In the case of strong drive the system does not reach a steady-state, and its evolution in time is calculated. To this end we considered the condensation model with an initially empty lattice and studied how the local density evolves in time. As long as the bulk dynamics does not permit condensation ($b < 2$) the growth of the local density is algebraic in time with exponents that we determined using a random walk picture for the boundary region, and standard hydrodynamic description in the bulk. From this analysis we are led to the conclusion that in the condensation model with $b > 2$ (where condensation in a periodic system sets in above the critical bulk density $\rho_c = 1/(b - 2)$) only the boundary sites develop into a condensate, with a density increasing linearly in time. All bulk sites become stationary with finite local fugacities determined by the boundary rates. Somewhat surprisingly the driven and the symmetric model have the same bulk and boundary growth exponents. The boundary condensate appears also for $b < 2$ when no condensation transition exists in a periodic system. It is a general feature of zero-range processes in which Z has a finite radius of convergence.

For $b > 1$ we observe a precursor to the condensation transition in the sense that the universal diffusive growth for $b < 1$ breaks down. Bulk and boundary growth exponents become different and both decrease with b , i.e. they become non-universal. It is interesting to note that $b = 1$ plays a special role also for the stationary state on a ring geometry. For $b < 1$ the stationary probability to find any given site empty vanishes as the density is increased to infinity,⁽¹⁹⁾ in agreement with intuition. However, for $b > 1$ every site has a finite probability of being empty, even if the particle density is infinite. Applied to present scenario this implies the counterintuitive result that even at very late time, when the average particle density in an

open system tends to infinity on *each site*, one still expects to find a finite fraction of empty sites at any given moment.

In this context it is also instructive to study the mean first passage time (MFPT) of the boundary particle density, i.e. the mean time τ after which a particle number N has been reached for the first time at a given site, starting from an empty site. Using the exact general MFPT expression⁽³⁹⁾ for the effective random walk defined above one finds

$$\tau = \frac{1}{b-1} \left[\binom{N+b}{b+1} - \binom{N+1}{2} \right], \quad (41)$$

where for non-integer b the factorials are defined by the Γ -function. For large N this quantity has the asymptotic behavior

$$\tau \sim \begin{cases} N^2 & (b < 1), \\ N^2 \ln N & (b = 1), \\ N^{1+b} & (b > 1). \end{cases} \quad (42)$$

Again there is a transition at $b=1$, with diffusive behavior for $b < 1$ and sub-diffusive exploration of the state space for $b > 1$. In the bulk a simple random picture for the on-site density dynamics is not valid and a prediction for the bulk MFPT is not possible. For small finite system size one expects boundary behavior everywhere, but with increasing corrections to scaling as one moves away from the boundary. The precise nature of the crossover from the boundary growth exponent to the hydrodynamic bulk growth exponent remains an open problem.

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REFERENCES

1. F. Spitzer, *Adv. Math.* **5**:246 (1970).
2. M. R. Evans, *Europhys. Lett.* **36**:13–18 (1996).
3. J. Krug and P. A. Ferrari, *J. Phys. A: Math. Gen.* **29**:L465–L471 (1996).
4. O. J. O’Loan, M. R. Evans, and M. E. Cates, *Phys. Rev. E* **58**:1404 (1998).
5. M. R. Evans, *Braz. J. Phys.* **30**:42 (2000).
6. I. Jeon and P. March, *Can. Math. Soc. Conf. Proc.* **26**:233–244 (2000).
7. G. M. Schütz, *J. Phys. A* **36**:R339 (2003).

8. G. M. Shim, B. Y. Park, J. D. Noh, and H. Lee, *Phys. Rev. E* **70**:031305 (2004).
9. Z. Burda, D. Johnston, J. Jurkiewicz, M. Kamiński, M. A. Nowak, G. Papp, and I. Zahed, *Phys. Rev. E* **65**:026102 (2002).
10. H. Fröhlich, *Phys. Lett. A* **51**:21–22 (1975).
11. D. Chowdhury, L. Santen, and A. Schadschneider, *Phys. Rep.* **329**:199 (2000); D. Helbing, *Rev. Mod. Phys.* **73**:1067 (2001).
12. G. Bianconi and A.-L. Barabási, *Phys. Rev. Lett.* **86**:5632 (2001).
13. S. N. Dorogovtsev, J. F. F. Mendes, and A. Samukhin, *Nucl. Phys. B* **666**:396 (2003).
14. Y. Kafri, E. Levine, D. Mukamel, G. M. Schütz, and J. Török, *Phys. Rev. Lett.* **89**:035702 (2002).
15. E. D. Andjel, *Ann. Probab.* **10**:525 (1982).
16. C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, Berlin, 1999).
17. S. Grosskinsky and H. Spohn, *Bull. Braz. Math. Soc. New Ser.* **34**:489 (2003).
18. I. Jeon and P. March, *Ann. Probab.* **28**:1162–1194 (2000).
19. S. Grosskinsky, G. M. Schütz, and H. Spohn, *J. Stat. Phys.* **113**:389 (2003)
20. C. Godrèche, *J. Phys. A: Math. Gen.* **36**:6313–6328 (2003).
21. S. N. Majumdar, S. Krishnamurthy, and M. Barma, *Phys. Rev. Lett.* **81**:3891 (1998).
22. B. Derrida, *Phys. Rep.* **301**:65 (1998).
23. G. M. Schütz, in *Phase Transitions and Critical Phenomena*, C. Domb and J. Lebowitz, eds., vol 19, (London: Academic, 2001).
24. A. M. Povolotsky, *Phys. Rev. E* **69**:061109 (2004).
25. G. M. Schütz, R. Ramaswamy, and M. Barma, *J. Phys. A: Math. Gen.* **29**:837 (1996).
26. M. Alimohammadi, V. Karimipour, and M. Khorrami, *Phys. Rev. E* **57**:6370 (1998).
27. S. Katz, J. L. Lebowitz, and H. Spohn, *J. Stat. Phys.* **34**:497 (1984).
28. T. Antal and G. M. Schütz, *Phys. Rev. E* **62**:83 (2000).
29. E. Levine, G. Ziv, L. Gray and D. Mukamel, *Physica A* **340**:636 (2004).
30. A. M. Povolotsky and J. F. F. Mendes, cond-mat/0411558.
31. V. B. Priezhev, E. V. Ivashkevich, A. M. Povolotsky, and C. K. Hu, *Phys. Rev. Lett.* **87**:084301 (2001).
32. G. Schönherr, *Phys. Rev. E* **71**:026122 (2005).
33. A. B. Kolomeisky, G. M. Schütz, E. B. Kolomeisky and J. P. Straley, *J. Phys. A: Math. Gen.* **31**:6911 (1998).
34. A. De Masi and P. Ferrari, *J. Stat. Phys.* **36**:81 (1984).
35. L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, *J. Stat. Phys.* **107**:635 (2002).
36. E. Levine, D. Mukamel, and G. M. Schütz, *Europhys. Lett.* **70**:565 (2005).
37. H. Spohn, *Large Scale Dynamics of Interacting Particle Systems* (Springer, Berlin, 1991).
38. E. Saada, in *Oberwolfach Reports* **1**:2266 (2004).
39. K. P. N. Murthy and K. W. Kehr, *Phys. Rev. A* **40**:2082 (1989); **41**:1160 (1990) (Erratum).